Periodic Solutions of Functional Differential Equations (Qualitative Theory of Solutions in Functional Differential Equations)

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Periodic solutions of functional differential equations

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1. Introduction. The purpose of this paper is to discuss the existence problem of periodic solutions of functional differential equations. Particularly, we are concerned with a reduction of a given ω-periodic functional differential equation with the delay $r > \omega$ to an auxiliary ω-periodic functional differential equation with the delay ω. Moreover, we obtain a Razumikhin type theorem concerning the existence of ω-periodic solutions of functional differential equations by using a strongly convex Liapunov function.

Let R^+ and R denote the interval $0 \leqq t < \infty$ and $-\infty < t < \infty$, respectively. For a given r, $0 < r \leqq \infty$, C_r denotes the Banach space of continuous and bounded functions defined by

$$C_r = \{ \phi : [-r, 0] \to R^n, \text{ continuous } \}, \quad 0 < r < \infty,$$

or

$$C_\infty = \{ \phi : (-\infty, 0] \to R^n, \text{ continuous and bounded } \},$$

with the uniform norm, $|\phi| = \sup \{ |\phi(\theta)|: -r < \theta \leqq 0 \}$, where $|\cdot|$ denotes the usual Euclidean norm in R^n. For a given continuous function $x(s)$, the symbol x_t will denote the element of C_r such that $x_t(\theta) = x(t+\theta)$, $-r < \theta \leqq 0$.

Let $f(t, \phi) : R \times C_r \to R^n$ be a completely continuous function which is ω-periodic in t, that is, $f(t+\omega, \phi) = f(t, \phi)$ for all $(t, \phi) \in R \times C_r$ and some positive constant ω.

Consider a functional differential equation

$$(1) \qquad \dot{x}(t) = f(t, x_t),$$

where $\dot{}$ denotes the right hand derivative.

2. Reductions of Equation (1). In this section, we shall consider reductions of Equation (1) with $r > \omega$ to auxiliary ω-periodic functional differential equations with the delay ω. Let $\sigma_t(\psi) : R \times C_\omega \to C_r$ be a mapping such that $\sigma_t(\psi)$ is ω-periodic in t, continuous on $R \times C_\omega$, takes bounded sets in $R \times C_\omega$ into bounded sets, $\sigma_t(\psi)(\theta)$ is ω-periodic in θ on $(-r, 0]$ if $\psi(0) = \psi(-\omega)$, and that $\sigma_t(\psi)(\theta)$ is ω-periodic in θ on $(-r, -\omega]$ if $r > 2\omega$. A simple example is :

$$(2) \quad \sigma(\psi)(\theta) = \begin{cases} \psi(\theta) \\ \psi(\theta+k\omega) - \dfrac{\theta+(k+1)\omega}{\omega}(\psi(0)-\psi(-\omega)), & -\min\{r, (k+1)\omega\} \leqq \theta < -k\omega, k \geqq 1. \end{cases}$$

For such a mapping $\sigma_t(\psi)$, let $g(t, \psi) : R \times C_\omega \to R^n$ be a function defined by

$$(3) \qquad g(t, \psi) = f(t, \sigma_t(\psi)).$$

Clearly $g(t, \psi)$ is completely continuous, and ω-periodic in t. For $g(t, \psi)$ defined by (3), consider an auxiliary equation

$$(4) \qquad \dot{x}(t) = g(t, x_t).$$

Then this equation always has a solution for the initial value problem, while Equation (1) may fail to have a solution for some initial value problem (see Seifert [3]). And we have the following theorem.

Theorem 1. An ω-periodic solution $x(t)$ ($t \in R$) of Equation (1) is a solution of Equation (4), and vice versa.

The proof of this theorem is clear from the definition of $g(t, \psi)$ and the properties of $\sigma_t(\psi)$.

3. Existence of periodic solutions of Equation (1). In this section, we shall show the existence of ω-periodic solutions of (1) via the existence of ω-periodic solutions of (4). Here we consider a strongly convex Liapunov function defined by strengthening the conditions for a convex Liapunov function in [2].

A function $V(t, x) : R \times R^n \to R$ is said to be a Liapunov function if $V(t, x)$ is continuous on $R \times R^n$ and satisfies $V(t, x) \geqq a(|x|)$ for a continuous function $a(u)$ such that $a(u) \to \infty$ as $u \to \infty$. We shall say a Liapunov function $V(t, x)$ is convex if for each fixed $t \in R$ the set $\{ x \in R^n : V(t, x) \leqq k\}$ is convex in R^n. Moreover, a convex Liapunov function $V(t, x)$ is said to be strongly convex if for each fixed $t \in R$ the set $\{ x \in R^n : V(t, x) = k\}$ is the boundary of $\{ x \in R^n : V(t, x) \leqq k\}$. We define $V'_{(1)}(t, \phi)$ by

$$V'_{(1)}(t, \phi) = \limsup_{\tau \to 0+} \frac{1}{\tau} \{V(t+\tau, x(t+\tau, t, \phi)) - V(t, \phi(0))\}.$$

Clearly we have

$$(5)\quad V'_{(1)}(t, \phi) = \limsup_{\tau \to 0+} \frac{1}{\tau} \{V(t+\tau, \phi(0)+\tau f(t, \phi)) - V(t, \phi(0))\}$$

if $V(t, x)$ is locally Lipschitzian with respect to x.

Now consider the following scalar equation

$$(6)\qquad \dot{u} = h(t, u),$$

where $h(t, u) : R \times R \to R$ is continuous and locally Lipschitzian with respect to u.

The following theorem is given in [1], where the delay r is finite.

Theorem 2. Let $V : R \times R^n \to R^+$ be a continuous, ω-periodic, convex Liapunov function, and let $L : R \to R$ be a continuous, non-decreasing function such that

$$(7)\qquad L(u) > u \text{ for all } u > 0,$$

or

$$(8)\qquad L(u) = u \text{ for all } u > 0.$$

Suppose that there exists a continuous function $u(t)$ defined on $[t_0-r, t_0+\omega]$ for some t_0 such that $u(t)$ is a solution of (6) on $[t_0, t_0+\omega]$ which satisfies $u(t_0+\omega+\theta) \leqq u(t_0+\theta)$ for $-r \leqq \theta \leqq 0$, $u(t) \geqq V(t, 0)$ for $t_0 \leqq t \leqq t_0+\omega$, and $u(t+\theta) \leqq L(u(t))$ for $t_0 \leqq t \leqq t_0+\omega$, $-r \leqq \theta \leqq 0$, and that we have

$$(9)\qquad V'_{(1)}(t, \phi) \leqq h(t, V(t, \phi(0)))$$

for all functions $\phi \in C_r$ with the property that

(10) $V(t, \phi(0)) \geqq u(t)$, $V(t+\theta, \phi(\theta)) \leqq L(V(t, \phi(0)))$ for $-r \leqq \theta \leqq 0$.

Then Equation (1) has an ω-periodic solution.

Now we can obtain the following theorem concerning the existence of an ω-periodic solution of Equation (1) with $r = \infty$ by combining Theorem 1 and Theorem 2.

Theorem 3. Let $V : R \times R^n \to R^+$ be a continuous, ω-periodic in t, strongly convex Liapunov function which is locally Lipschitzian with respect to x, and let $L : R \to R$ be a nondecreasing continuous function which satisfies (7) or (8). Suppose that Equation (6) has a solution $u(t)$ on $[0, \omega]$ such that $u(0) \geqq u(\omega)$, $u(t) > V(t, 0)$ for $0 \leqq t \leqq \omega$, $\bar{u} \leqq L(\underline{u})$ where $\bar{u} = \max_{0 \leqq t \leqq \omega} u(t)$, $\underline{u} = \min_{0 \leqq t \leqq \omega} u(t)$, and that we have

(11) $$V'_{(1)}(t, \phi) \leqq h(t, V(t, \phi(0)))$$

for all functions $\phi \in C_\infty$ with the property that

(12) $\phi \in P_\infty$, $V(t, \phi(0)) \geqq u(t)$, $V(t+\theta, \phi(\theta)) \leqq L(V(t, \phi(0)))$ for $-\infty < \theta \leqq 0$,

where $P_\infty = \{ \phi \in C_\infty : \phi(\theta)$ is ω-periodic on $(-\infty, -\omega]\}$. Then Equation (1) has an ω-periodic solution $x(t)$ such that $V(t, x(t)) \leqq u(t)$ for $0 \leqq t \leqq \omega$.

This theorem can be proved in the following manner. First, we define a function $\bar{\sigma}_t(\psi) : R \times C_\omega \to C_\infty$ which is similar to $\sigma_t(\psi)$ in Section 2 by using the strongly convex Liapunov function $V(t, x)$. For any $(t, \psi) \in R \times C_\omega$, let $v = \max_{-\omega \leqq \theta \leqq 0} V(t+\theta, \psi(\theta))$ and

$v^* = \max\{v, L(\underline{u})\}$. Define a set Σ^* by

$$\Sigma^* = \{(s, x) \in R \times R^n : V(s, x) \leqq v^*\}.$$

Now let $\sigma(\psi)$ be a function defined by (2). For $\theta \leqq 0$ such that $(t+\theta, \sigma(\psi)(\theta)) \in \Sigma^*$, define $\bar{\sigma}_t(\psi)(\theta)$ by $\bar{\sigma}_t(\psi)(\theta) = \sigma(\psi)(\theta)$. For $\theta < -\omega$ such that $(t+\theta, \sigma(\psi)(\theta)) \notin \Sigma^*$, define $\bar{\sigma}_t(\psi)(\theta)$ by $\bar{\sigma}_t(\psi)(\theta) = \lambda(\theta)\sigma(\psi)(\theta)$, where $\lambda(\theta)$ is a uniquely determined number such that $\lambda(\theta) \in (0, 1)$ and $V(t+\theta, \lambda(\theta)\sigma(\psi)(\theta)) = v^*$. Clearly the function $\bar{\sigma}_t(\psi)$ has the same properties with $\sigma_t(\psi)$ in Section 2. Moreover, we have $V(t+\theta, \bar{\sigma}_t(\psi)(\theta)) \leqq v^*$ for $\theta \leqq 0$ by the definition of $\bar{\sigma}_t(\psi)$. Using this $\bar{\sigma}_t(\psi)$, define an auxiliary function $g^*(t, \psi)$ by

$$g^*(t, \psi) = f(t, \bar{\sigma}_t(\psi)), \tag{13}$$

and consider the following auxiliary equation

$$\dot{x}(t) = g^*(t, x_t). \tag{14}$$

If we show that

$$V'_{(14)}(t, \psi) \leqq h(t, V(t, \psi(0))) \tag{15}$$

under the condition

$$V(t, \psi(0)) \geqq u(t),\ V(t+\theta, \psi(\theta)) \leqq L(V(t, \psi(0))) \text{ for } -\omega \leqq \theta \leqq 0, \tag{16}$$

then we can apply Theorem 2 to Equation (14). Let (t, ψ) satisfy (16), and let $v = \max_{-\omega \leqq \theta \leqq 0} V(t+\theta, \psi(\theta))$ and $v^* = \max\{v, L(\underline{u})\}$. If we take $\phi = \bar{\sigma}_t(\psi)$ for (t, ψ), then we obtain

$$V(t+\theta, \phi(\theta)) \leqq v^* \leqq L(V(t, \phi(0))) \text{ for } -\infty < \theta \leqq 0.$$

Thus (16) implies (12), and consequently we have (11), which is not different from (15) by (5) and (13), since $V(t, x)$ is locally Lipschitzian with respect to x.

Next, $u(t)$ can be extended continuously on $[-\omega, \omega]$ by defining $u(t) = \max\{u(0), u(t+\omega)\}$ for $-\omega \leqq t \leqq 0$. Then all assumptions of Theorem 2 are satisfied with this $u(t)$ and $t_0 = 0$, and Equation (14) has an ω-periodic solution. Thus we can obtain the conclusion of this theorem by Theorem 1.

Finally, we present an application of Theorem 3. Consider a scalar ω-periodic functional differential equation

$$\text{(17)} \qquad \dot{x}(t) = -x(t) + 4x(t-\omega) - 4x(t-2\omega) + f(t, x_t),$$

where $f(t, \phi) : R \times C_\infty \to R$ is completely continuous, ω-periodic in t, and satisfies

$$\text{(18)} \qquad |f(t, \phi)| \leqq |\phi| \text{ if } t \in R, \ |\phi(0)| > K$$

for some positive constant K. If we take $L(u) = u$ and $V(x) = \frac{x^2}{2}$, then V is a strongly convex Liapunov function which is locally Lipschitzian, and we have

$$V'_{(17)}(t, \phi) = -\phi^2(0) + \phi(0)f(t, \phi) \leqq -\phi^2(0) + \phi^2(0)\frac{|\phi|}{|\phi(0)|}\frac{|f(t, \phi)|}{|\phi|} \leqq 0$$

under the condition (12) if the solution $u(t)$ of (6) with $h(t, u) \equiv 0$ satisfies $u(t) > \frac{K^2}{2}$, where K is the one in (18). Therefore, by Theorem 3, there exists an ω-periodic solution of (17).

Suppose that (17) is an equation with the finite delay r. In this case, we cannot apply Theorem 37.1 in [4] to (17) to conclude that (17) has an ω-periodic solution, since r is greater than the period ω. Moreover, $V'_{(17)}(t, \phi)$ cannot be compared with $h(t, V(t, \phi(0))) \equiv 0$ under the condition (10). Thus we have no idea how to apply Theorem2.

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